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A dimension formula for reduced plethysms

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Abstract
The boson calculus formalism is used to construct realizations of basis states of irreducible representations of unitary groups taking as a paradigm the interacting boson models of atomic nuclei. These realizations, together with a theorem on plethysms for obtaining branching rules, allowed us to obtain a dimension formula for reduced plethysms.

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1. Introduction

Due to the Pauli principle, the group of permutation on n objects, $S(n)$, plays an important role in the study of many-particle systems. According to this principle, in an n-particle system in which the states available for each particle have symmetry $\{\lambda\}$ under some subgroup of the general linear group $GL(n)$; these states must be classified as states with definite symmetry under $S(n)$. This problem can be solved using plethysm of Schur functions as done by Elliott [1] for the p and s, d-shells of the harmonic oscillator shell model and by us [2] for the p, f-shell.

Plethysm was introduced by Littlewood in his classical book [3]. A survey of properties and main results in plethysm can be found in [4–7], and more recently in [8].

Plethysms are also powerful tools to find branching rules for chains and lattices of groups. A systematic use of plethysm in atomic spectroscopy was done by Wybourne [4]. In [8], we used the plethysm technique to address the labelling problem of states in interacting boson models (IBMs) [9]. Also an application to the classification of states in the $O(8)$ pairing model will soon be available [10].

A fundamental problem of a plethysm is to expand it into terms which have a definite unitary symmetry as well as a symmetry with respect to the symmetric group. It is a difficult problem and many algorithms to solve it have appeared in the literature [12–19], beginning with a recursive method due to Murnaghan [11] used by Butler and Wybourne in [18]. These algorithms are complicated and a dimensional check of their results is desirable. To this end the dimension formula (8) given below can be used. However, in physical applications only some terms in the plethysm expansion are relevant. When only these terms are considered we call the resulting expansion a 'reduced plethysm'. The aim of this paper is to find a counterpart of formula (8) for the reduced plethysms.

In section 2 we will establish the notation used and give a few definitions and results in order to make this paper self contained. Sections 3–5 give intermediate results and in section 6 the final result is given.

2. Small survey of plethysms and related quantities

A (standard) partition is a sequence

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t, 0, 0, \ldots) \tag{1}$$

of non-negative integers λ_i in non-decreasing order

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_t > 0 \tag{2}$$

and containing only finitely many (t) non-negative *parts* λ_i. The value t is called its *length* and the sum of its parts, its *degree*. A partition of length t is said to be t rowed. Partitions are used to label irreducible representations (irreps) of symmetric ($[\lambda]$), unitary ($\{\lambda\}$) orthogonal ((λ)), symplectic ($\langle\lambda\rangle$), ..., groups and this terminology extends to these irreps.

Consider a fixed number n of variables $x_1, x_2, \ldots, x_n$. The Schur function in these variables, labelled by the partition λ, is defined by

$$\{\lambda\} = \frac{|M(\lambda)|}{|M(\lambda = 0)|} \tag{3}$$

where $M(\lambda)$ is a $n \times n$ matrix with elements $M_{ij} = (x_i)^{\lambda_j+n-j}$. For partitions with length greater than n, one sets $\{\lambda\} \equiv 0$. (Note that we use the same notation for Schur functions and irreps of unitary groups. When this notation becomes ambiguous we will denote Schur functions by s_λ.) The Schur function $\{\lambda\}$ is a homogeneous polynomial of degree equal to that of λ with non-negative integer coefficients in the variables x_i.

The product of two Schur functions $\{\lambda'\}$ and $\{\lambda''\}$ of degrees r' and r'' can be decomposed into a sum of Schur functions of degree $r' + r''$:

$$\{\lambda'\}\{\lambda''\} = \sum_\lambda \Gamma_{\lambda'\lambda''\lambda}\{\lambda\}. \tag{4}$$

The coefficients $\Gamma_{\lambda'\lambda''\lambda}$ are non-negative integers and give the multiplicity of the irreps $\{\lambda\}$ in the reduction of the Kronecker product of irreps $\{\lambda'\}$ and $\{\lambda''\}$ of unitary groups. The operation with two Schur functions defined by equation (4) is called the *outer product* of these Schur functions.

Another operation with two Schur functions, now of the same degree r, is the *inner product*

$$\{\lambda'\} \times \{\lambda''\} = \sum_\lambda g_{\lambda'\lambda''\lambda}\{\lambda\} \tag{5}$$

where the sum over λ extends to all partitions of r and the coefficients $g_{\lambda'\lambda''\lambda}$ are non-negative integers that give the multiplicity of the $S(r)$ irrep $[\lambda]$ in the expansion of the Kronecker product of irreps $[\lambda']$ and $[\lambda'']$.

The k-fold outer product of a Schur function $\{\lambda\}$ of degree r can be decomposed into terms $\{\lambda\} \otimes \{\mu\}$ with definite symmetry $[\mu]$ under $S(k)$ as

$$\{\lambda\}^k \equiv \{\lambda\}\{\lambda\}\cdots\{\lambda\} = \sum_\mu d_{[\mu]}\{\lambda\} \otimes \{\mu\} \tag{6}$$

where the sum runs over the partitions μ of k and the coefficients $d_{[\mu]}$ are the dimensions of the $S(k)$ irreps $[\mu]$. These terms are called *plethysms* of the Schur functions $\{\lambda\}$ and $\{\mu\}$ and obviously have degree kr. As a consequence they can be written as a linear combination of Schur functions $\{\nu\}$ whose degrees equal the product of the degree of $\{\lambda\}$ by the degree of $\{\mu\}$:

$$\{\lambda\} \otimes \{\mu\} = \sum_{\nu} \Lambda_{\lambda\mu\nu}\{\nu\}. \tag{7}$$

Littlewood proved that the coefficients $\Lambda_{\lambda\mu\nu}$ are non-negative integers. They satisfy the dimension formula

$$d_{\{\lambda\}\otimes\{\mu\}} \equiv \sum_{\nu} d_{[\nu]}\Lambda_{\lambda\mu\nu} = \frac{(rs)!}{(r!)^s s!}(d_{[\lambda]})^s d_{[\mu]}. \tag{8}$$

where r and s are the degrees of λ and μ.

Based on the procedure for obtaining a general irrep of a group G in terms of multiple Kronecker product of its defining irrep, the following theorem is obtained [4]. *If under the restriction $G \to H$ the character $\langle\!\langle 1\rangle\!\rangle$ of group G decomposes as*

$$\langle\!\langle 1\rangle\!\rangle = \langle\alpha\rangle + \langle\beta\rangle + \cdots + \langle\omega\rangle, \tag{9}$$

then the character $\langle\!\langle \lambda\rangle\!\rangle$ of G decomposes into the characters $\langle\rho\rangle$ of H according to the characters contained in the plethysm

$$[\langle\alpha\rangle + \langle\beta\rangle + \cdots + \langle\omega\rangle] \otimes \langle\!\langle \lambda\rangle\!\rangle\ . \tag{10}$$

This plethysm can be obtained expressing the characters of G and H in terms of the characters of $GL(n)$, evaluating the resulting plethysms of $GL(n)$ characters and re-expressing the result in terms of the characters of H. Using the association *irrep* $\leftrightarrow$ *character*, this theorem gives us the coefficients of the reduction of the irrep $\langle\!\langle \lambda\rangle\!\rangle$ of G into direct sum of irreps $\langle\rho\rangle$ of H.

To use this theorem it is fundamental to know how the irrep $\langle\!\langle 1\rangle\!\rangle$ of G decomposes into irreps of H. In this paper this task is achieved by use of the *boson calculus* [20] to find realizations of $\langle\!\langle 1\rangle\!\rangle$. As a paradigm of this procedure we use the interacting boson models formalism.

3. Reduced plethysms

For small values of the degrees of the Schur functions, plethysms could be computed by hand using, for example, equation (6.23) in [5] or a recursive method due to Murnaghan [11]. However, for total degree $\gtrsim$10, hand computing becomes impossible. With the advent of computers, the computation of more complex plethysms became feasible and efforts were made in order to find new algorithms for their computation [8], [13–17]. If the total degree is not very large, these algorithms work very well and for total degrees $\lesssim$40, the plethysm computation is done within reasonable memory and time allocations. However, for total degrees $\gtrsim$50, such as in applications to medium-heavy nuclei, computer codes take a very long time and require very large memory space. Another observed characteristic is the extremely large multiplicities of the Schur functions that appear in the expansion of the plethysms. Fortunately, G and H, being subgroups of $GL(n)$, have finite ranks and normally, instead of *complete* plethysms, only 'reduced plethysms' are needed in applications. By reduced plethysm we mean the part of a plethysm that contains only Schur functions with length up to

Table 1. Comparison of number of terms and largest multiplicities for some $k = 2$-, 3-, 4-reduced plethysms and the complete plethysm. The first two columns contain the factors of plethysms. For each plethysm the first line under the remaining columns gives the number of terms of the (k-reduced) plethysm and the second, its largest multiplicity.

$\{\mu\}$	$\{\lambda\}$	$k = 2$	$k = 3$	$k = 4$	Complete
{2}	{10}	6	14	23	42
		1	1	1	1
{21}	{8}	1	27	106	1 160
		1	4	38	484
{4}	{9}	17	118	452	5 674
		5	38	183	504
{22}	{9}	1	12	178	5 674
		1	3	38	25 868
{31}	{72}	7	95	418	15 883
		2	2068	208 672	36 590 125
{211}	{54}	1	12	178	15 920
		1	1	38	56 951 359
{31}	{4311}	0	77	401	16 145
		0	8714	1456 584	109 649 416
{31}	{621}	5	91	418	16 100
		1	6169	765 289	107 502 037

a given value, say k. Denoting the reduced plethysms by $\{\lambda\} \overset{k}{\otimes} \{\mu\}$, one has

$$\{\lambda\} \overset{k}{\otimes} \{\mu\} = \sum_{\nu} \Lambda^{(k)}_{\lambda\mu\nu}\{\nu\} \equiv \text{only the Schur functions of } \{\lambda\} \otimes \{\mu\} \text{ with length up to } k. \tag{11}$$

When using plethysms in physical applications, one is tempted to compute complete plethysms and reduce them to our needs. Instead of that it is more efficient to start with reduced plethysms from the beginning.

In [8] we have presented an algorithm for computing plethysms that works also for reduced plethysms. Such an algorithm uses as a starting point a formula for $\{n\} \otimes \{m\}$ (given by equation (13) in [13]) that works for $\{n\} \overset{k}{\otimes} \{m\}$ too. There are also other algorithms [15, 21] which can be adapted to obtain reduced plethysms. By use of conjugation properties, reduced plethysms can be applied in situations in which the irreps we are interested in are restricted to maximum k columns. This happens when the groups of interest are complementary to some low rank groups and a most recent example here is the reduction for the group chains in the $O(8)$ pairing model for heavy $N \sim Z$ nuclei [22].

The computing time for calculating reduced plethysms is drastically reduced when compared with the time needed for complete plethysms. In addition, reduced plethysms contain terms with smaller multiplicities. In table 1 we present the number of terms and the greatest multiplicities for reduced plethysms as compared to those of the complete plethysms. A problem arises when one tries to verify the plethysm results. For complete plethysms, dimension formula (8) holds . However, for reduced plethysms no dimension formula is available in the literature. This formula will be obtained in the following sections where use of the plethysm results for the interacting boson models of atomic nuclei will be made in order to stress the relevance of reduced plethysms in physical applications.

4. Interacting boson models with $SU(3)$ symmetry, unitary groups and reduced plethysms

In the IBM with sd bosons (called sdIBM or IBM-1), the building blocks are the creation $\left(s^\dagger, d_\mu^\dagger\right)$ and annihilation (s, d_μ) boson operators. In a compact notation, one can write

$$b_\rho^\dagger \qquad \text{with} \qquad b_\rho^\dagger = d_{\rho-3}^\dagger \qquad \text{for} \qquad \rho = 1, 2, 3, 4, 5 \quad \text{and} \quad b_6^\dagger = s^\dagger. \tag{12}$$

These operators satisfy the usual commutation relations

$$[b_\rho, b_{\rho'}] = \left[b_\rho^\dagger, b_{\rho'}^\dagger\right] = 0, \qquad \left[b_\rho, b_{\rho'}^\dagger\right] = \delta_{\rho\rho'}. \tag{13}$$

With these operators one can construct operators $\mathcal{C}_\rho^{\rho'}$ that are generators of $U(6)$, the spectrum generating algebra (SGA) of sdIBM. Special linear combinations of these operators give the generators of $O^+(6)$, $U(5)$, $O^+(5)$, $SU(3)$, $O^+(3)$ and $O^+(2)$ algebras that appear in the three group–subgroup chains of sdIBM,

$$\begin{array}{ccccccccccl} & \nearrow & U(5) & \supset & O^+(5) & \supset & O^+(3) & \supset & O^+(2) & \text{(I)}, \\ U(6) & \rightarrow & SU(3) & \supset & O^+(3) & \supset & O^+(2) & & & \text{(II)}, \\ & \searrow & O^+(6) & \supset & O^+(5) & \supset & O^+(3) & \supset & O^+(2) & \text{(III)}. \end{array} \tag{14}$$

For chain (II), branching rules for the reduction $U(6) \rightarrow U(3)[SU(3)]$ are required for labelling the basis states. Here, the basic association (see equation (9)) for the irreps is

$$\{1\}_{U(6)} = \{2\}_{U(3)}. \tag{15}$$

Then, using equation (10) it is seen easily that a general $U(6)$ irrep $\{\lambda\}$ decomposes into $U(3)$ irreps $\{\mu\}$ according to

$$\{\lambda\} = \sum_\mu \Lambda_{\{2\}\lambda\mu}^{(3)} \{\mu\}. \tag{16}$$

In equation (16) the expansion coefficients of reduced plethysm $\{2\} \overset{3}{\otimes} \{\lambda\}$ appear because $\{\mu\}$, being an $U(3)$ irrep cannot have length greater than 3. A dimension analysis of equation (16) yields

$$\sum_\mu \Lambda_{\{2\}\lambda\mu}^{(3)} \dim(\{\mu\}_3) \equiv \dim(\{2\} \overset{3}{\otimes} \{\lambda\}) = \dim(\{\lambda\}_6) \tag{17}$$

where the notation $\{\nu\}_n$ means that the irrep $\{\nu\}$ must be taken as a $U(n)$ irrep. Equation (17) provides a dimension formula for reduced plethysms $\{2\} \overset{3}{\otimes} \{\lambda\}$.

A similar situation occurs in sdgIBM (also called IBM-1G) which differs from sdIBM by the introduction of a new hexadecupole degree of freedom with $\ell = 4$ for the bosons. Then one has boson creation operators $b_{\ell m}^\dagger$ with $\ell = 0, 2, 4, -\ell \leqslant m \leqslant \ell$ and the SGA here is $U(15)$. The sdgIBM also admits $SU(3)$ subalgebra and for this the basic association is $\{1\}_{U(15)} = \{4\}_{U(3)}$. The branching rules for the $U(15) \rightarrow U(3)$ irrep reductions are then given by

$$\{\lambda\} = \sum_\mu \Lambda_{\{4\}\lambda\mu}^{(3)} \{\mu\} \tag{18}$$

and tabulations for symmetric irreps $\{m\}$ with $m \leqslant 15$ are available in the literature [8, 21]. Taking the dimensions on both sides of equation (18), one obtains the formula

$$\sum_{\mu} \Lambda^{(3)}_{\{4\}\lambda\mu} \dim(\{\mu\}_3) \equiv \dim(\{4\} \overset{3}{\otimes} \{\lambda\}) = \dim(\{\lambda\}_{15}) \tag{19}$$

for reduced plethysms $\{4\} \overset{3}{\otimes} \{\lambda\}$. Generalizing this reasoning, one can think of a general IBM model having bosons with $\ell = n, n-2, \ldots, 0$ or 1; for example $\ell = 1, 3$ gives the IBM with p and f bosons which is a part of the $sdpf$ and $sdgpf$ IBMs [23]. This gives a unitary group of dimension

$$\sum_{\ell=n,n-2,\ldots,0or;1} (2\ell+1) = (n+1)(n+2)/2 \equiv \dim(\{n\}_3). \tag{20}$$

Its irrep $\{1\}$ will contain all the ℓs of the nth energy level of the three-dimensional harmonic oscillator. As is known, their wavefunctions span the $U(3)$ irrep $\{n\}$ with dimension given by equation (20). We would have the basic association $\{1\}_{(n+1)(n+2)/2} = \{n\}_3$ and then $\{\lambda\} = \sum_{\mu} \Lambda^{(3)}_{\{n\}\lambda\mu}\{\mu\}$ providing the dimension formula

$$\sum_{\mu} \Lambda^{(3)}_{\{n\}\lambda\mu} \dim(\{\mu\}_3) \equiv \dim(\{n\} \overset{3}{\otimes} \{\lambda\}) = \dim\big(\{\lambda\}_{\dim(\{n\}_3)}\big) \tag{21}$$

for left symmetric 3-reduced plethysms.

5. More general reduced plethysms from interacting boson models

Considering a generalized IBM in which the building blocks are boson operators having all the ℓ values of a given $U(3)$ irrep $\{\mu\}$, we will have

$$\{1\}_{\dim(\{\mu\}_3)} = \{\mu\}_3 \tag{22}$$

leading to the dimension formula

$$\sum_{\nu} \Lambda^{(3)}_{\{\mu\}\lambda\nu} \dim(\{\nu\}_3) \equiv \dim(\{\mu\} \overset{3}{\otimes} \{\lambda\} = \dim\big(\{\lambda\}_{\dim(\{\mu\}_3)}\big) \tag{23}$$

for *all* 3-reduced plethysms.

Extended IBMs such as sdgIBM, $sdpf$IBM, etc will allow for going beyond the three-rowed irreps. For example, sdgIBM gives $U(15) \supset U(5)$ with the basic association $\{1\}_{15} = \{2\}_5$ and the $U(15)$ irreps $\{\lambda\}$ decompose into $U(5)$ irreps according to

$$\{\lambda\} = \sum_{\mu} \Lambda^{(5)}_{\{2\}\lambda\mu}\{\mu\} \tag{24}$$

giving the dimension formula

$$\sum_{\mu} \Lambda^{(5)}_{\{2\}\lambda\mu} \dim(\{\mu\}_5) \equiv \dim(\{2\} \overset{5}{\otimes} \{\lambda\}) = \dim(\{\lambda\}_{15}). \tag{25}$$

Similarly $U(15) \supset U(6)$ with the basic association $\{1\}_{15} = \{1^2\}_6$ gives the dimension formula

$$\sum_{\mu} \Lambda^{(6)}_{\{1^2\}\lambda\mu} \dim(\{\mu\}_6) \equiv \dim(\{1^2\} \overset{6}{\otimes} \{\lambda\}) = \dim(\{\lambda\}_{15}). \tag{26}$$

Equations (25) and (26) are explicitly verified for $\{m\}_{15}$ for $m \leqslant 15$ in the tabulations given in the last reference of [21]. With $sdpf$IBM in the pf sector one has $U(10) \supset U(4)$ with $\{1\}_{10} = \{2\}_4$ and $U(10) \supset U(5)$ with $\{1\}_{10} = \{1^2\}_5$. They give the dimension formulae

$$\begin{aligned} \sum_{\mu} \Lambda^{(4)}_{\{2\}\lambda\mu} \dim(\{\mu\}_4) &\equiv \dim(\{2\} \overset{4}{\otimes} \{\lambda\}) = \dim(\{\lambda\}_{10}) \\ \sum_{\mu} \Lambda^{(5)}_{\{1^2\}\lambda\mu} \dim(\{\mu\}_5) &\equiv \dim(\{1^2\} \overset{5}{\otimes} \{\lambda\}) = \dim(\{\lambda\}_{10}). \end{aligned} \tag{27}$$

6. Final formula and comments

Up to now we have considered three-dimensional coordinate space in detail and some examples from four-, five- and six-dimensional spaces. Considering, instead, a general k-dimensional coordinate space, the same reasoning used in deriving equations (17), (19), (21), (23), (25)–(27) will give the final formula

$$\sum_{\nu} \Lambda^{(k)}_{\{\mu\}\lambda\nu} \dim(\{\nu\}_k) \equiv \dim(\{\mu\} \overset{k}{\otimes} \{\lambda\} = \dim\big(\{\lambda\}_{\dim(\{\mu\}_k)}\big) \tag{28}$$

for general k-reduced plethysms. The physical meaning of equation (28) is as follows. The plethysm $\{\mu\} \otimes \{\lambda\}$ restricted to k rows can be viewed, from the $U(N) \supset U(k)$ irrep reduction problem, as generating the reductions of the irrep $\{\lambda\}$ of $U(N)$ to those of $U(k)$ given the basic association $\{1\}_N = \{\mu\}_k$. Then the irreps given by the plethysm cannot have more than k rows as they must belong to $U(k)$. Secondly, it is clear that N is the dimension of the irrep $\{\mu\}$ of $U(k)$. These two observations immediately give the formula (28) for reduced plethysms.

In conclusion a formula for the dimensions of k-reduced plethysms is derived starting from the known results for various interacting boson models of atomic nuclei.

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